ELASTIC-WAVE DIFFRACTION BY A FINITE CRACK UNDER ANTIPLANE STRAIN CONDITIONS

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The problem of elastic-wave interaction with a crack has been examined in [1, 2] for normal wave incidence in the case of plane or antiplane strain. The solution of the problem about arbitrary wave incidence in the plane strain case is described in [3].

The diffraction of a step stress wave by a finite rectilinear crack at an aribtrary angle of incidence in the case of antiplane strain is considered in Sec. 1 of this paper. The mathematical description of the motion of an elastic medium is simpler for antiplane than for plane strain. This permits obtaining simpler and more complete solutions.

The diffraction by a crack for normal wave incidence under the condition that the crack reaches the critical state at some time and starts to develop under the effect of the incident wave is considered in Sec. 2 in the same formulation.

1. Oblique Incidence of a Stress Wave on a Fixed Crack

An infinite elastic solid containing an isolated crack of length $2l$ is considered. The strain state is assumed antiplanar. The stress-tensor components differ from zero,

$$\tau_{xy} = \mu\partial w/\partial x, \qquad \tau_{zy} = \mu\partial w/\partial z \tag{1.1}$$

where $w = w(x, z, t)$ is the single nonzero component of the displacement vector. Let us introduce the dimensionless variables

$$x' = \frac{x}{l}, \quad z' = \frac{z}{l}, \quad w' = \frac{w}{l}, \quad \tau' = \frac{\tau}{\mu}, \quad t' = t\frac{c}{l}, \quad c = \sqrt{\mu/\rho}$$

where c is the transverse wave velocity, and μ is the shear modulus. The Equations (1.1) and the equation of motion for the dimensionless quantities are written as (for simplicity in the writing the primes are omitted)

$$\tau_{xy} = \frac{\partial w}{\partial x}, \quad \tau_{zy} = \frac{\partial w}{\partial z}, \quad \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial z^2} - \frac{\partial^2 w}{\partial t^2} = 0$$

Let a shear wave in the form of a step with amplitude p_0 be incident on the crack from the left, and let it be tangent to the left edge of the crack at the point $x = -1$ at the time $t = 0$. The angle between the direction of the wave-front normal and the positive direction of the z axis will be denoted by φ. The problem about the incidence of such a wave on a crack is equivalent to the problem about a free crack on whose edges the following loading appears at the initial time:

$$p(x, t) = \mp\tau_{yz} = \pm p_0 \cos\varphi\, H[t - (x + 1)\sin\varphi] \tag{1.2}$$
$$H(\zeta) = \begin{cases} 1, & \zeta > 0 \\ 0, & \zeta < 0 \end{cases}$$

The upper sign in the equality refers to the upper edge of the slit. The crack length remains constant the whole time.

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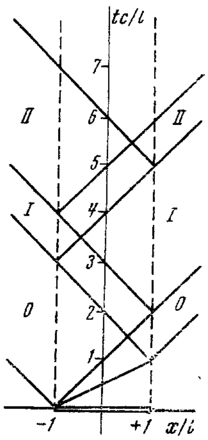

Fig. 1

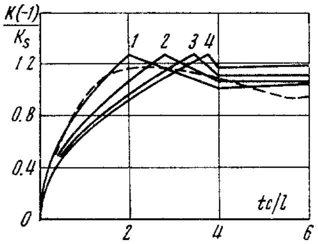

Fig. 2

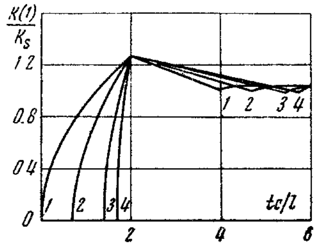

Fig. 3

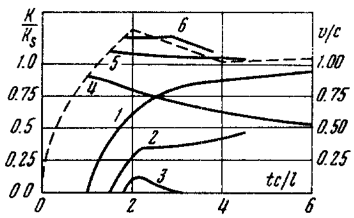

Fig. 4

The form presented for $p(x, t)$ means that the load is propagated at the intensity $\pm p_0 \cos\varphi$ along the crack length from the edge $x = -1$ at the constant velocity $(\sin\varphi)^{-1}$ until it reaches the opposite edge of the crack. After this, the load remains constant. The solution of the diffraction problem can be obtained by adding the solution of the problem of a free crack subjected to the load (1.2) at the initial instant to the solution describing the motion of an incident wave in a continuous infinite elastic body. For such a wave

$$w(x, z, t) = -p_0 H[t - (1 + x)\sin\varphi - z\cos\varphi]$$

In such a wave the stress-tensor component τ_{yz} will be

$$\tau_{yz} = p_0 \cos\varphi\, H[t - (1 + x)\sin\varphi]$$

on the crack $z = 0, -1 < x < +1$.

In combination with the value from (1.2), this yields the condition $\tau_{yz} = 0$, on the crack, which should indeed be satisfied in the diffraction problem. The problem about the load (1.2) is a particular case of the problem solved in [4]. Represented in Fig. 1 is the kinematics of diverse wave fronts which originate because of wave reflection from the crack edges. The solution is sought in the zones denoted by the numbers 0, I, II.

According to [4], the expressions for τ_{yz} in zone 0 are for $z = 0$

$$\tau_{10}(\xi_0, \eta_0) = \frac{p_0 \cos\varphi}{\pi \sqrt{\eta_0 - \eta_1(\xi_0)}} \int_{-\gamma\xi_0+\beta}^{\eta_1(\xi_0)} \frac{\sqrt{\eta_1(\xi_0) - \eta}}{\eta_0 - \eta}\, d\eta$$

$$\tau_{20}(\xi_0, \eta_0) = \frac{p_0 \cos\varphi}{\pi \sqrt{\xi_0 - \xi_1(\eta_0)}} \int_{-\gamma_1\eta_0+\beta_1}^{\xi_1(\eta_0)} \frac{\sqrt{\xi_1(\eta_0) - \xi}}{\xi_0 - \xi}\, d\xi \tag{1.3}$$

$$\xi = (t - x)/\sqrt{2}, \quad \eta = (t + x)/\sqrt{2}, \quad \xi_1 = \sqrt{2} + \eta, \quad \eta_1 = \sqrt{2} + \xi$$

$$\gamma = \frac{1 + \sin\varphi}{1 - \sin\varphi}, \quad \gamma_1 = \gamma^{-1}, \quad \beta = \frac{\sqrt{2}\sin\varphi}{1 - \sin\varphi}, \quad \beta_1 = \frac{\sqrt{2}\sin\varphi}{1 + \sin\varphi}$$

where ξ, η are characteristic variables: τ_{10}, τ_{20} are the values of τ_{yz} for the right and left ends of the crack, respectively. Evaluating the integrals in (1.3), we obtain

$$\tau_{10}(\xi, \eta) = \frac{2p_0 \cos\varphi}{\pi} \left\{ \left[\frac{2\xi + \sqrt{2}(1 - 2\sin\varphi)}{(\eta - \xi - \sqrt{2})(1 - \sin\varphi)} \right]^{1/2} - \operatorname{arctg} \left[\frac{2\xi + \sqrt{2}(1 - 2\sin\varphi)}{(\eta - \xi - \sqrt{2})(1 - \sin\varphi)} \right]^{1/2} \right\}$$

$$\tau_{20}(\xi, \eta) = \frac{2p_0 \cos\varphi}{\pi} \left\{ \left[\frac{2\eta + \sqrt{2}}{(\xi - \eta - \sqrt{2})(1 + \sin\varphi)} \right]^{1/2} - \operatorname{arctg} \left[\frac{2\eta + \sqrt{2}}{(\xi - \eta - \sqrt{2})(1 + \sin\varphi)} \right]^{1/2} \right\} \tag{1.4}$$

As is seen from (1.3), (1.4), τ_{10} and τ_{20} have singularities of order $(\Delta x)^{-1/2}$, where $\Delta x \ll 1$, as η_0 tends to η_1 (ξ_0 to ξ_1). In the limit case we can write

$$\tau_{10}(\xi_0, \eta_1) = \frac{K_{10}(\xi_0)}{\sqrt{\Delta x}} = \frac{1}{\sqrt{\Delta x}} \left[\frac{1}{\pi 2^{1/4}} \int_{-\gamma\xi_0+\beta}^{\eta_1(\xi_0)} \frac{\tau(\xi_0, \eta)}{\sqrt{\eta_1(\xi_0) - \eta}}\, d\eta \right] \tag{1.5}$$

$$\tau_{20}(\xi_1, \eta_0) = \frac{K_{20}(\eta_0)}{\sqrt{\Delta x}} = \frac{1}{\sqrt{\Delta x}} \left[\frac{1}{\pi 2^{1/4}} \int_{-\gamma_1\eta_0+\beta_1}^{\xi_1(\eta_0)} \frac{\tau(\xi, \eta_0)}{\sqrt{\xi_1(\eta_0) - \xi}}\, d\xi \right]$$

$$K_{10}(\xi) = \frac{p_0}{\pi} 2^{3/4} [2\xi + \sqrt{2}(1 - 2\sin\varphi)]^{1/2} \cos\beta$$
$$K_{20}(\eta) = \frac{p_0}{\pi} 2^{3/4} [2\eta + \sqrt{2}]^{1/2} \sin\beta \quad \left(\beta = \frac{\pi}{4} - \frac{\varphi}{2}\right) \tag{1.6}$$

Here K_{10} and K_{20} are intensity factors for the singularities at the heads of the cracks approached from outside. The quantities K_{10} and K_{20} are monotonically increasing functions of their arguments, where up to the time of arrival of the first reflected wave from the opposite end of the crack, i.e., for K_{10} at $\xi = 1/\sqrt{2}$, and for K_{20} at $\eta = (1 + 2\sin\varphi)/\sqrt{2}$, their values agree and are equal to $(p_0/\pi)\, 2^{3/2} \sin 2\beta$. Hence, it is seen that as φ varies between 0 and $\pi/2$, the maximum value of K_{10} and K_{20} varies between $(p_0/\pi)\, 2^{3/2}$ and 0. For normal wave incidence on the crack, i.e., for $\varphi = 0$, τ_{10} agrees with τ_{20}.

The stress-intensity coefficients in zones I and II are computed from the formulas

$$K_{11}(\xi_0) = \frac{2^{-1/4}}{\pi} \left\{ - \int_{-1/\sqrt{2}}^{\eta_2} \frac{\tau_{20}(\xi_0, \eta)}{\sqrt{\eta_1(\xi_0) - \eta}} d\eta + 2^{7/4} p_0 \cos\varphi \right\}, \quad \frac{1}{\sqrt{2}} \leqslant \xi_0 < \frac{3 + 2\sin\varphi}{\sqrt{2}}$$

$$K_{12}(\xi_0) = \frac{2^{-1/4}}{\pi} \left\{ - \int_{-1/\sqrt{2}}^{\eta_*} \frac{\tau_{20}(\xi_0, \eta)}{\sqrt{\eta_1(\xi_0) - \eta}} d\eta - \int_{\eta_*}^{\eta_2} \frac{\tau_{21}(\xi_0, \eta)}{\sqrt{\eta_1(\xi_0) - \eta}} d\eta + 2^{7/4} p_0 \cos\varphi \right\}, \quad \frac{3 + 2\sin\varphi}{\sqrt{2}} < \xi_0 < \frac{5}{\sqrt{2}}$$

$$K_{21}(\eta_0) = \frac{2^{-1/4}}{\pi} \left\{ - \int_{\xi_*}^{\xi_2} \frac{\tau_{10}(\xi, \eta_0)}{\sqrt{\xi_1(\eta_0) - \xi}} d\xi + 2^{7/4} p_0 \cos\varphi \right\}, \quad \frac{1 + 2\sin\varphi}{\sqrt{2}} \leqslant \eta_0 < \frac{3}{\sqrt{2}}$$

$$K_{22}(\eta_0) = \frac{2^{-1/4}}{\pi} \left\{ - \int_{\xi_*}^{1/\sqrt{2}} \frac{\tau_{10}(\xi, \eta_0)}{\sqrt{\xi_1(\eta_0) - \xi}} d\xi - \int_{1/\sqrt{2}}^{\xi_2} \frac{\tau_{11}(\xi, \eta_0)}{\sqrt{\xi_1(\eta_0) - \xi}} d\xi + 2^{7/4} p_0 \cos\varphi \right\}, \quad \frac{3}{\sqrt{2}} < \eta_0 < \frac{5 + 2\sin\varphi}{\sqrt{2}} \tag{1.7}$$

$$\tau_{11}(\xi_0, \eta_0) = \frac{1}{\pi\sqrt{\eta_0 - \eta_1}} \left\{ - \int_{-1/\sqrt{2}}^{\eta_2} \tau_{20}(\xi_0, \eta) \frac{\sqrt{\eta_1 - \eta}}{\eta_0 - \eta} d\eta + p_0 \cos\varphi \int_{\eta_2}^{\eta_1} \frac{\sqrt{\eta_1 - \eta}}{\eta_0 - \eta} d\eta \right\}$$

$$\tau_{21}(\xi_0, \eta_0) = \frac{1}{\pi\sqrt{\xi_0 - \xi_1}} \left\{ - \int_{\xi_*}^{\xi_2} \tau_{10}(\xi, \eta_0) \frac{\sqrt{\xi_1 - \xi}}{\xi_0 - \xi} d\xi + p_0 \cos\varphi \int_{\xi_2}^{\xi_1} \frac{\sqrt{\xi_1 - \xi}}{\xi_0 - \xi} d\xi \right\}$$

$$\eta_2 = -\sqrt{2} + \xi, \quad \xi_2 = -\sqrt{2} + \eta, \quad \xi_* = \frac{(2\sin\varphi - 1)}{\sqrt{2}}, \quad \eta_* = \frac{(2\sin\varphi + 1)}{\sqrt{2}}$$

The first subscript in these expressions indicates the end of the crack. Thus, 1 corresponds to the right end x=+1, and 2 to the left end x=−1; the second subscript denotes the zone or the quantity of arrived reflected waves. The expressions (1.7) were evaluated numerically. The results of a numerical computation of the time change in the ratio between the stress-intensity factor and its static value are presented in Figs. 2 and 3. The curves in Fig. 2 correspond to the left end x=−1, and in Fig. 3 to the right end x=1. The curves denoted by the numbers 1, 2, 3, and 4 correspond to the values $\varphi = 0$, $\pi/8$, $\pi/4$, $\pi/3$, respectively. It is seen from these graphs that the stress-intensity factors, referred to their static value, first rise to 1.27, then flucuate and tend to one. This agrees with the solutions in [1-3]. Shown for comparison in Fig. 2 by dashes is the solution for the normally incident wave taken from [2]. The difference from the curve 1 corresponding to normal incidence can be explained by the error in the approximate calculations used to obtain the solution in [1].

From the fracture viewpoint, both ends of the crack are equally dangerous for step-type waves. The maximal stresses originate earlier at the far end of the crack (x=1), therefore, fracture can indeed start there.

2. Diffraction by a Developing Crack

If the stresses which appear during wave diffraction by a crack exceed the critical value, then the crack starts to grow. The law of motion of the crack end is determined by the equation [4]

$$\dot{x} = \frac{(r^4 - 8\pi T^2)}{(r^4 + 8\pi T^2)}, \quad r = \frac{\pi 2^{1/4} K}{\sqrt{1 - \dot{x}}} \tag{2.1}$$

Here T is the energy lost irreversibly as the end of the crack advances per unit length; K is the stress-intensity factor at the tip of the crack, which is determined by formulas analogous to those presented in Sec. 1 for a fixed crack.

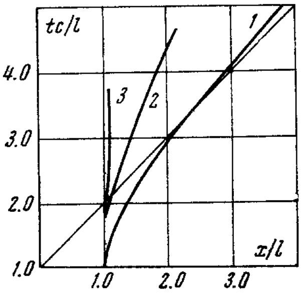

Fig. 5

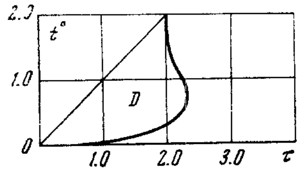

Fig. 6

It is henceforth assumed that T = const. The case of normal wave incidence on a crack is considered. The motion of the crack starts at the time $t = t_0$ at which the stress intensity reaches its critical value and $t^2_0 = \pi T^2/4p_0^4$. By virtue of symmetry, the consideration can be limited to the motion of one end of the crack. The law of right-end motion is written as

$$x(t) = 1 + t + [\pi/2 - 1 - 2 \operatorname{arc\,tg} t / t_0]\, t_0 \tag{2.2}$$

which has been found in [4] for a semiinfinite crack. This law is valid for finite cracks even to the time of wave arrival from its opposite end. The magnitude of this time t_1 can be determined,

$$t_1 = t_0 \operatorname{tg} [1/t_0 + \pi/4 - {}^1/_2] \tag{2.3}$$

For $t_0 = 4/(2+\pi)$ this expression becomes infinite, i.e., for $t_0 \le 4/(2+\pi)$ the wave from the left end of the crack cannot reach the right end. In this case, the law (2.2) is satisfied all the time that the stress in the incident wave is held at the initial level p_0. If $t_0 > 4/(2+\pi)$, then for $t > t_0 \tan [1/t_0 + \pi/4 - {}^1/_2]$ the influence of the left end of the crack starts to be felt. Investigating the dependence of the crack acceleration on t_0, it can be established that for $t_0 < 1.71$ the motion of the crack tip continues to be accelerated after the first diffraction, but for $t_0 > 1.71$ it starts to slow down and can even cease.

Presented in Fig. 4 are graphs of the time change in crack-development rate for $t_0 = 1.0$, 1.5, 1.8 (curves 1, 2, and 3, respectively), obtained by numerical integration of (2.1). The law of time variation of the stress-intensity factor for the presented values of t_0 is shown in Fig. 4 by curves numbered 4, 5, and 6. The stress-intensity factor K is referred to its static value $K_C = p_0\sqrt{l/2}$, where $2l$ is the initial length of the crack. The dashed line in this graph corresponds to the case of a fixed crack. The trajectories of crack motion for $t_0 = 1.0$, 1.5, 1.8 are shown in Fig. 5 by the curves 1, 2, and 3. The straight line in Fig. 5 is the trajectory of a wave emanating from the left end of the crack.

Let us present some estimates of the influence of the wave duration on the finite size of the crack. Let us assume that a rectangular wave of duration τ is normally incident on the crack. The final size of the crack is determined by the parameters t_0 and τ in this case.

If $t_0 > 2$, which means that the stress-intensity factor does not reach its critical value, then the crack does not generally move from its place. Similarly for $\tau \le t_0$. These rest domains are shown by dashes in the $t_0\tau$ plane in Fig. 6.* The line in Fig. 6 bounds the domain D of the states $t_0\tau$ for which the stopping of the crack will occur before the first diffraction. The law of crack motion for this domain after the load has been removed is described for $t = \tau$ by the equation

$$\dot{x} = [(\sqrt{t} - \sqrt{t-\tau})^4 - t_0^2]\,[(\sqrt{t} - \sqrt{t-\tau})^4 + t_0^2]^{-1} \tag{2.4}$$

Stopping of the crack occurs at the time $t_f = (\tau + t_0)^2/4t_0$. The increment in the crack length after removal of the load at $t = \tau$ until the time of stopping equals

$$\Delta x = \frac{t_0^2 + \tau^2}{4t_0}\left(\operatorname{arc\,tg}\frac{\tau}{t_0} - \frac{\pi}{4}\right) - \frac{\tau^2 - t_0^2}{4t_0}$$

The equation of the boundary curve in Fig. 6 is represented as

$$(4 + \tau)\, t_0 - 2t^2_0 - \tau^2 + (\tau^2 - 3t_0^2)(\operatorname{arc\,tg} \tau/t_0 - \pi/4) = 0$$

The solution has not been obtained successfully in final form outside the domain mentioned. It can only be noted that for large τ the final size of the crack will be on the order of τ, since the limiting velocity of the crack equals one.

The assumption of the constancy of T is essential for the results obtained. In some experimental researches on the normal tension of optically active polymers [5] it was detected that T grows as the crack grows. As a result, it turns out that the stress-intensity factor is also a growing function of the time, and the limiting velocity of crack propagation is considerably less than the theoretical limit, the Rayleigh value.

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* As in Russian original. The dashes do not appear in Fig. 6 — Publisher.

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